

Knowing Mathematics Well Enough to Teach It

Mathematical Knowledge for Teaching (MKT)

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Mathematical Knowledge for Teaching (MKT)

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Overview of session

1. Understanding the question
2. Mathematical knowledge for teaching (MKT): A program of research
3. Discussion: Approaches to developing teachers' MKT

What is "enough" mathematical knowledge to be an effective teacher?

- A major in mathematics?
- At the pre-secondary school level, majoring in mathematics is not associated with greater gains in students' learning
- At the secondary level, uneven results, not powerful

How would you explain these results?

By early 1990s . . .

1. Many studies claiming that U.S. teachers lacked mathematical knowledge (but sampling problems)
2. Evidence that course-taking or majoring in math did not provide "deep" or "flexible" knowledge, nor that it was correlated with student achievement gains
3. Same items used across studies, but with little or no investigation of their psychometric properties

Getting the question right

- How much mathematics do teachers need to know?
- What mathematics do teachers need to know, and why?
- What mathematical knowledge and skill is entailed by teaching?

Taking a step back: What mathematical knowledge and skill is involved in *teaching* mathematics?

Starting with practice: What mathematics does teaching entail?

1. Study instruction and identify the mathematical work of teaching
2. Analyze what mathematical knowledge is entailed by the work (MKT)
3. Test the working hypotheses based on these analyses by developing measures of MKT, validating teacher scores against practice and against student achievement gains
4. Develop and evaluate approaches to helping teachers learn mathematical knowledge for teaching

BIG CLUE:

Understanding that teaching mathematics is a specialized kind of mathematical work

A problem in teaching

You ask your learners to explain what a rectangle is. One child offers a definition:

> A rectangle is a flat shape. It has four square corners, and it is closed all the way around.

What is the work for the teacher?

A rectangle is a flat shape. It has four square corners, and it is closed all the way around.

1. To see that something is missing
2. Decide what to do or say
3. Offer a counterexample

What is a shape that satisfies this definition and yet is not a rectangle?

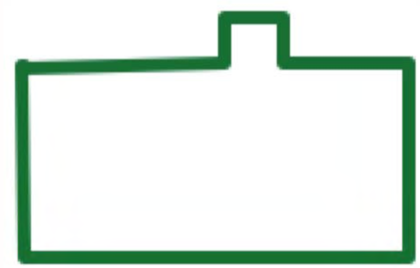
What is the work for the teacher?

A rectangle is a flat shape. It has four square corners, and it is closed all the way around.

1. To see that something is missing
2. Decide what to do or say
3. Offer a counterexample

"Is this a rectangle?"

(straight sides)



(exactly four square corners)

A rectangle is a flat shape with **straight sides that are connected at exactly** four square corners. It is closed all the way around.

Teaching involves special kinds of mathematical work

1. Solving special kinds of mathematical problems
2. Engaging in specialized mathematical reasoning
3. Using mathematical language precisely but accessibly

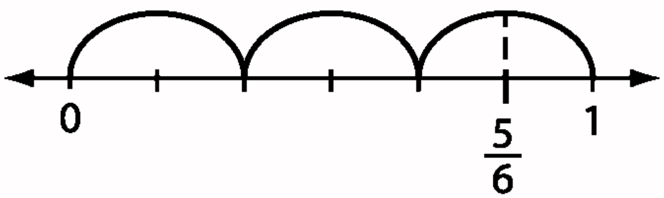
Choosing pedagogically-strategic examples

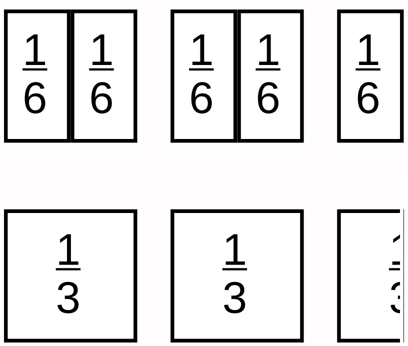
Which of the following lists would be best for assessing whether your students understand decimal ordering? Justify your choice.

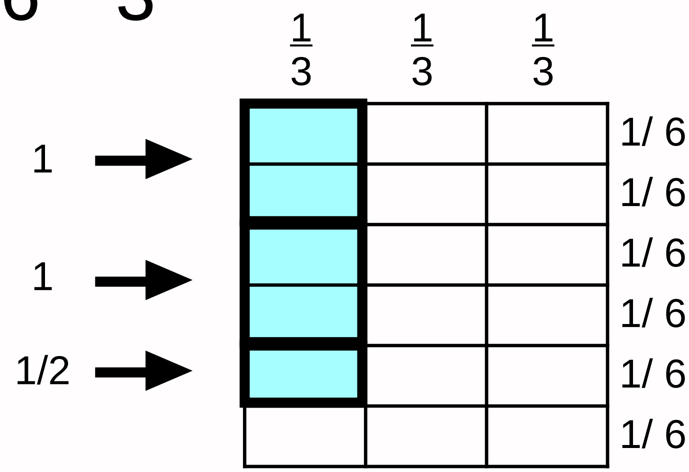
A.	**.5**	**7**	**.01**	**11.4**
B.	**.60**	**2.53**	**3.12**	**.45**
C.	**.6**	**4.25**	**.565**	**2.5**

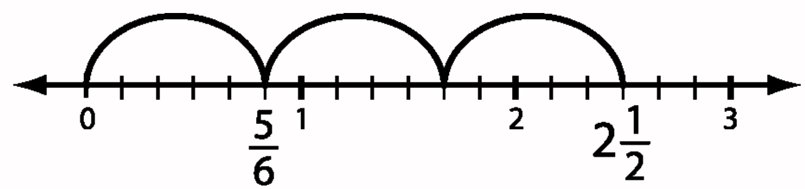
Analyzing — and “talking” — representations

Which of these can be used to represent $\frac{5}{6} \div \frac{1}{3}$? Explain with reference to all parts of the expression.









Analyzing errors

What mathematical steps could have produced this answer?

(a)

```
   49
x  25
-----
  405
108
-----
 1485
```

(b)

```
   49
x  25
-----
  225
  100
-----
  325
```

(c)

```
   49
x  25
-----
 1250
   25
-----
 1275
```

Analyzing non-standard (but correct) responses

Which student is using a method that could be used to multiply any two whole numbers?

Student A	Student B	Student C
35 x 25 1 25 +7 5 8 75	35 x25 175 +700 875	35 x 25 25 1 50 1 00 + 600 875

Using language precisely and accessibly #1

A polygon is a simple closed plane curve composed of finitely many straight line segments.

OUR DEFINITIONS

Polygon:

Using language precisely and accessibly #2

a) An even number is a number that can be divided into two equal parts.
b) An even number is any multiple of 2.
c) An even number is any integer multiple of 2.
d) An even number is any number whose unit digit is 0, 2, 4, 6, or 8.
e) A whole number is even if it is the sum of a whole number with itself.

Are these definitions okay? What are the consequences of any of these?

a) An even number is a number that can be divided into two equal parts.

b) An even number is any multiple of 2.

All numbers, for example 7, 3/5, √2, π, are even!

c) An even number is any integer multiple of 2.

This is a correct definition of even number.

d) An even number is any number whose unit digit is 0, 2, 4, 6, or 8.

In this case, 36.7 is an even number!

e) A whole number is even if it is the sum of a whole number with itself.

This is a correct definition of evenness for whole numbers, and is consistent with the general definition for integers that will arrive later.

Posing mathematics problems

POSSIBLE PROBLEM:

I have pennies, nickels, and dimes in my pocket. If I pull out two coins, what amounts of money might I have?

Reasoning about different wording

How does the exact wording of the problem affect the mathematical work for my students? Which best fits my goals for them?

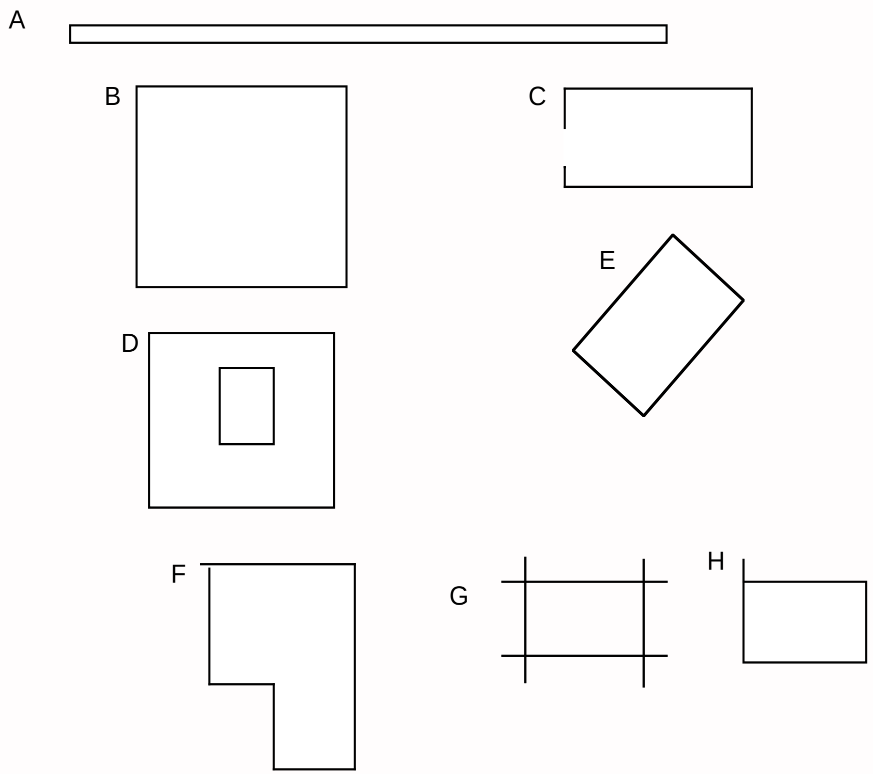
Reasoning about different wording

1. I have pennies, nickels, and dimes in my pocket. If I pull out two coins, what amount of money might I have?

2. I have pennies, nickels, and dimes in my pocket. If I pull out two coins, how many combinations are possible?

3. I have pennies, nickels, and dimes in my pocket. If I pull out two coins, how many different amounts of money are possible? Prove that you have found all the amounts that are possible.

Reasoning about starting points and sequencing



- In a whole-class discussion aimed at developing the concept and definition for “rectangle,” which figure would be good to discuss first? Why?
- How would you sequence these figures to develop the concept and definition of rectangle? Why?

Teaching involves special kinds of mathematical work

1. Solving special kinds of mathematical problems
2. Engaging in specialized mathematical reasoning
3. Using mathematical language precisely but accessibly

1. Solving special kinds of mathematical problems

- Selecting or constructing a strategic example, representation, or task
- Analyzing representations, definitions, non-standard correct responses, incorrect responses
- Developing and using appropriate language

2. Engaging in specialized mathematical reasoning

- Comparing mathematical alternatives
- Justifying choices of representations, language, tasks
- Reasoning about starting points and sequences

3. Using mathematical language precisely but accessibly

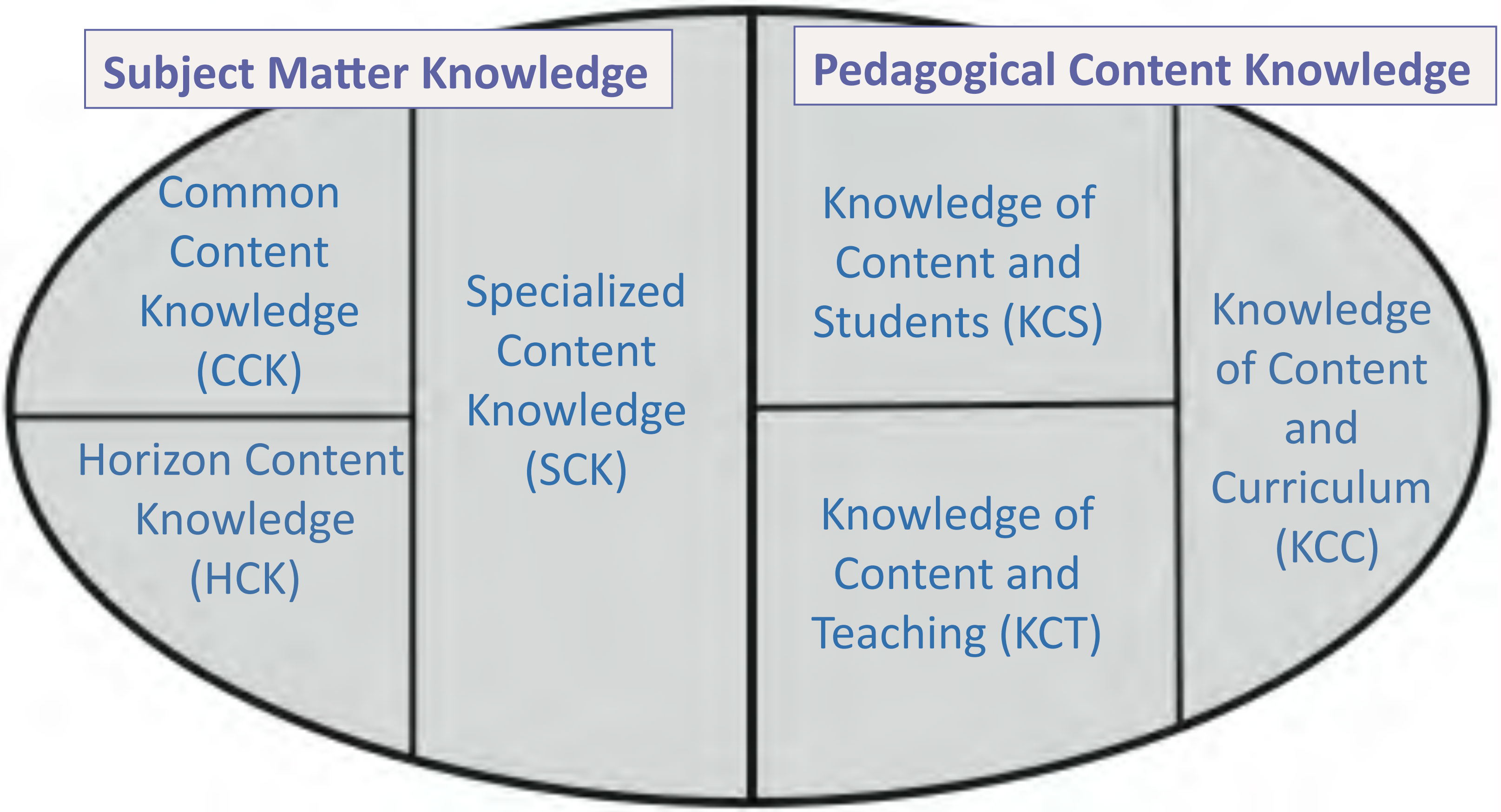
- Coordinating between mathematical details and precision and terms children know or can learn to use
- Coordinating between simple, invented, or everyday language and mathematical details
- Judging what can be left more casual and what not

What makes this *special* mathematical work?

1. It is performed in the service of helping others learn mathematics.
2. Its warrants are tied both to pedagogical purpose and mathematical integrity.

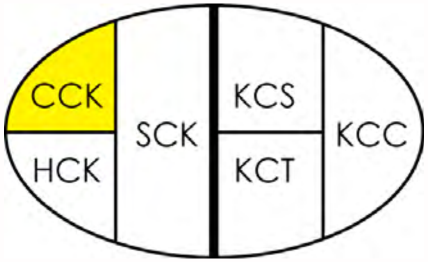
What mathematical resources are needed to engage in these special kinds of mathematical work?

Mathematical Knowledge for Teaching (MKT)



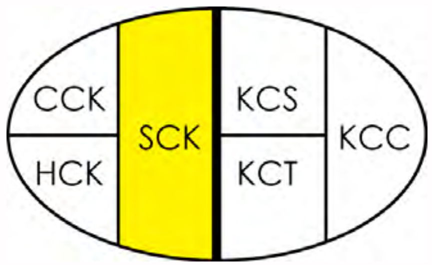
A sequence to illustrate domains of MKT

Concept of a rectangle



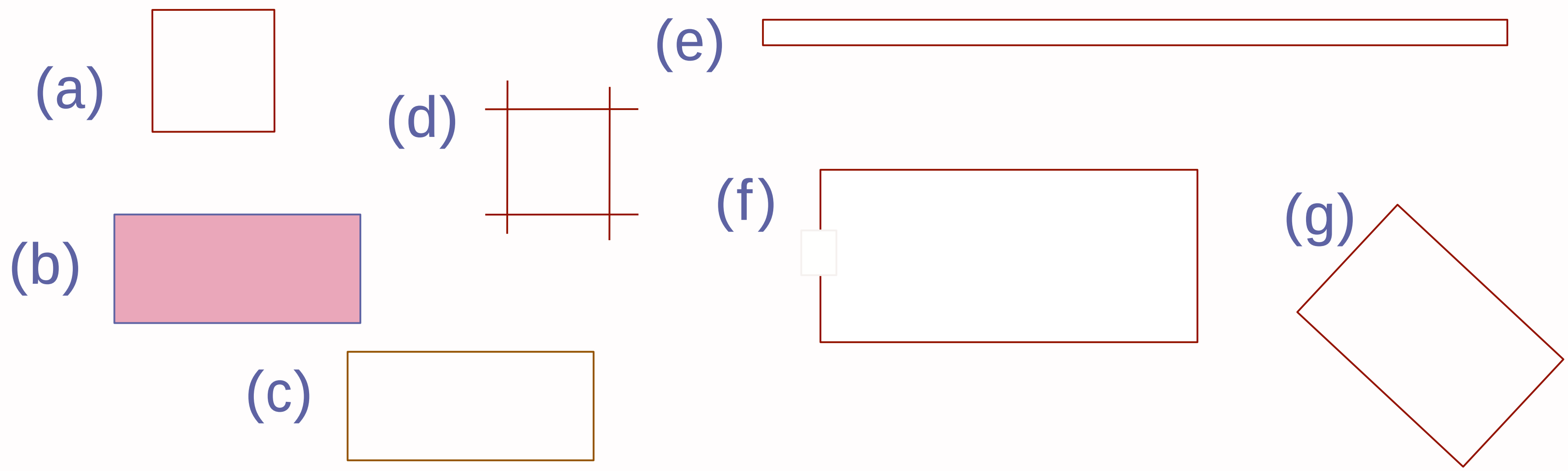
Common content knowledge (CCK)

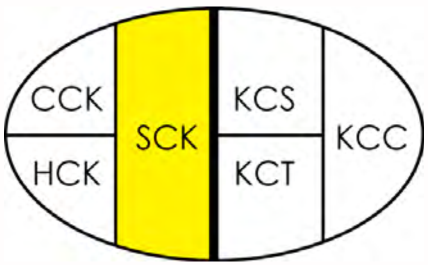
Draw a **rectangle.**



Specialized content knowledge (SCK)

Which of these figures would be good to present to assess whether students understand what a rectangle is, and <u>why</u>?





Specialized content knowledge (SCK)

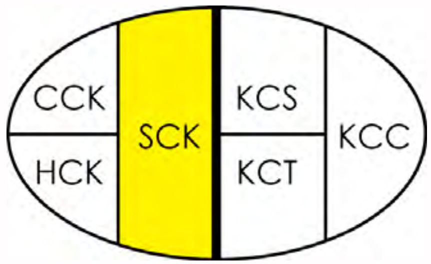
Which of these is a mathematically accurate definition of "rectangle"?

① A rectangle is a figure with four straight sides, two long and two shorter.

② A rectangle is a shape with exactly four connected straight line segments meeting at right angles.

③ A rectangle is flat, and has four straight line segments, four square corners, and it is closed all the way around.

For any that are <u>not</u> mathematically accurate, give an example that shows what is wrong.



Specialized content knowledge (SCK)

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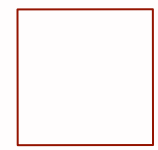
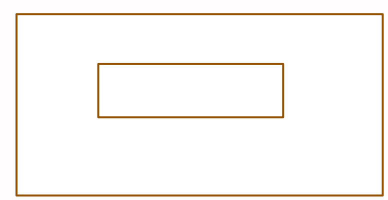
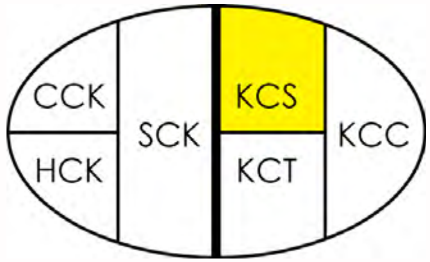


② A rectangle is a shape with exactly four connected straight line segments meeting at right angles.

③ A rectangle is flat, and has four straight line segments, four square corners, and it is closed all the way around.

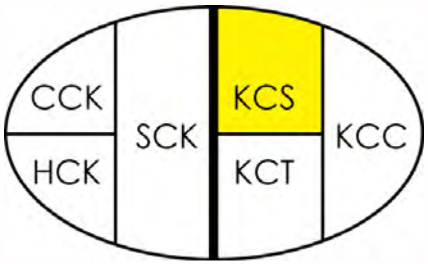




Knowledge of content and students (KCS)

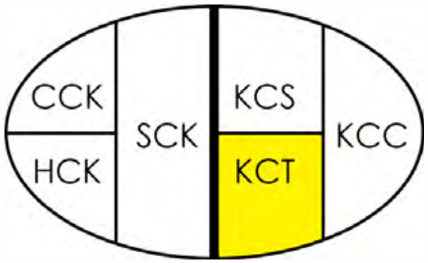
- Write a mathematically accurate definition of "rectangle" that is usable by second graders.

- How can the notion of "simple closed curve" be expressed in a way that is both mathematically accurate and usable? Which part of this phrase is most challenging for children?

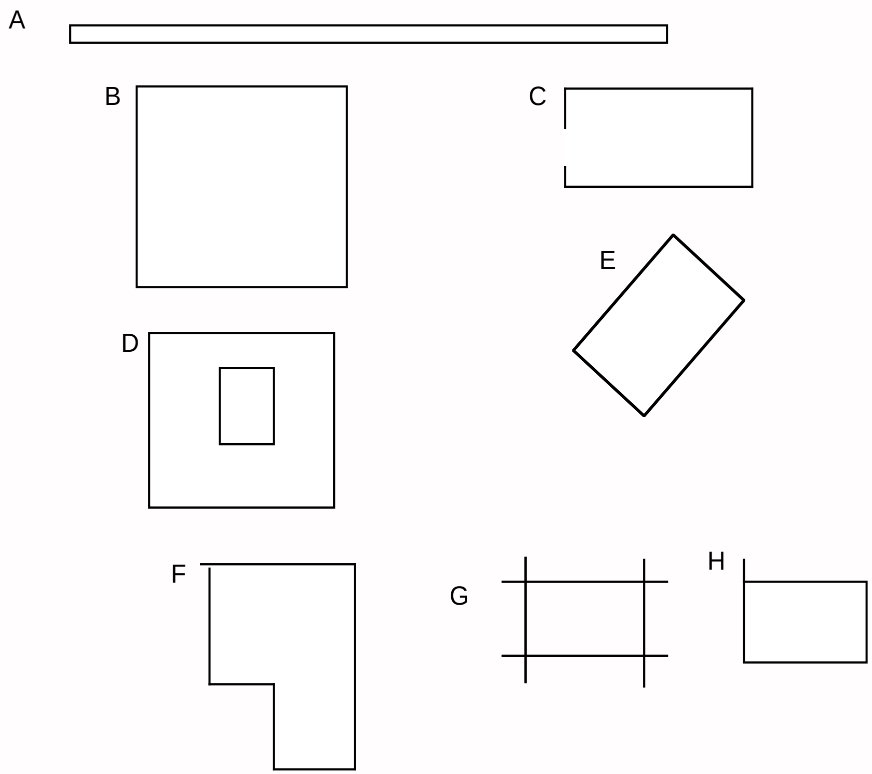


Knowledge of content and students (KCS)

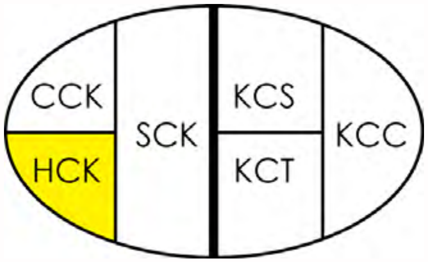
- What are students likely to know about what a rectangle is?
- What do students typically have difficulty with in learning about rectangles, and why?



Knowledge of teaching and content (KCT)



- How would you sequence these figures to discuss the concept of a rectangle?
- What task would you create using these figures (or others) to set up a productive discussion aimed at developing a definition?
- In a whole-class discussion, which one would be good to discuss first?



Horizon content knowledge (HCK)

- Is it okay to shade in the figures shown to students – e.g., 

- What are the issues involved with the fact that children learn about rectangles before polygons?

What is MKT not?

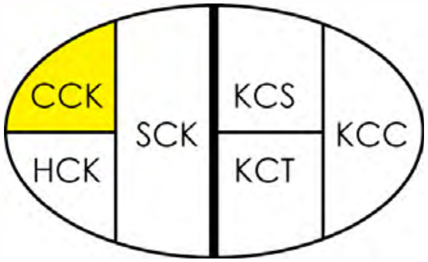
1. It is not just knowing the topics of the school curriculum.
2. It is not knowing school content from an "advanced perspective."
3. It is not just another term for pedagogical content knowledge.
4. It is not being able to teach the content.

What is MKT?

1. Knowing mathematics from the perspective of helping others learn it.
2. Being mathematically ready to teach an idea, method, or other aspect of maths.

A second sequence to illustrate domains of MKT

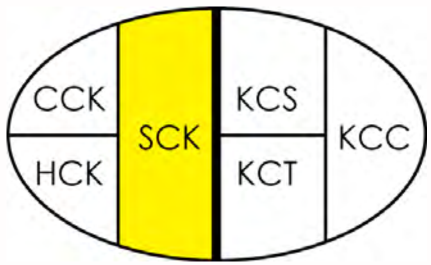
Division of fractions



Common content knowledge (CCK)

Calculate:

$$\frac{5}{6} \div \frac{1}{3}$$



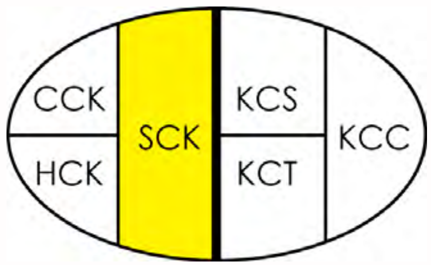
Specialized content knowledge (SCK)

$$\frac{5}{6} \div \frac{1}{3} = \frac{10}{12} \div \frac{4}{12} = 10 \div 4 = 2\frac{1}{2}$$

Is this a fluke?

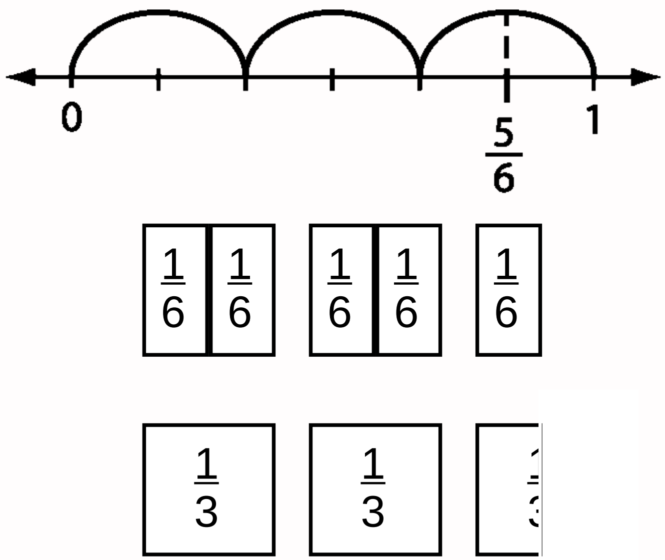
Does it work in general?

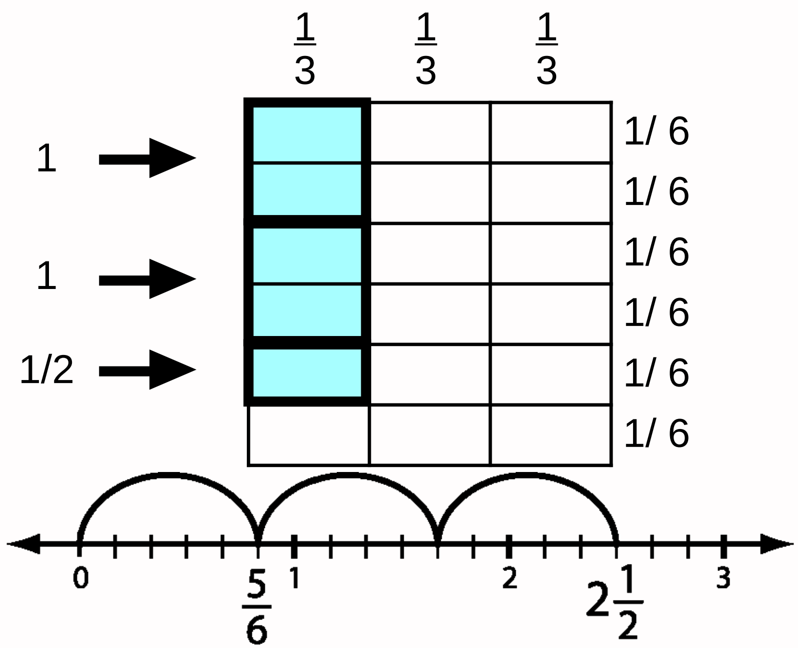
If so, why does it work?

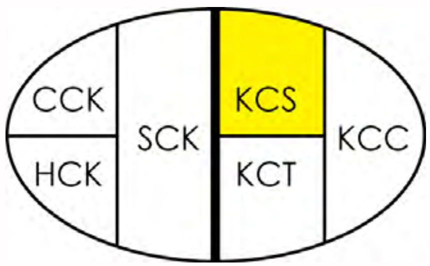


Specialized content knowledge (SCK)

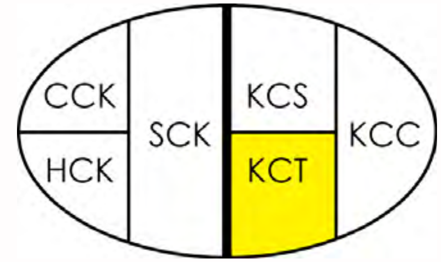
Which of these can be used to represent $\frac{5}{6} \div \frac{1}{3}$?







$$\frac{5}{6} \div \frac{1}{3} = 2\frac{1}{2}$$

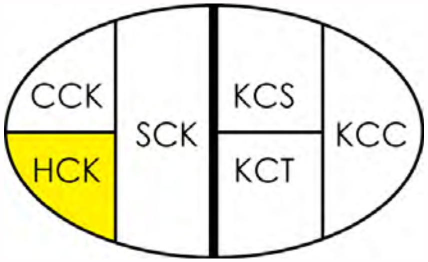


Knowledge of students and content (KCS)

- What are common errors students make when dividing fractions?
- How do students' experiences with division of whole numbers support their understanding of division of fractions? How does it confuse them?
- What difficulties do students typically have interpreting the answer to a division of fractions problem?

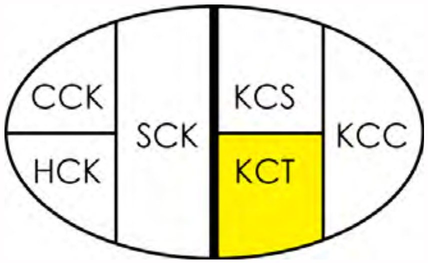
Knowledge of teaching and content (KCT)

- Which representation would you use to introduce the meaning of division of fractions? Or to explain the invert and multiply algorithm?
- What sequence of problems would you use to begin work on division of fractions?
- In a whole-class discussion, what solution methods would you want presented, and in what order?



Horizon content knowledge (HCK)

- A student comments that "if you divide by smaller and smaller fractions, the answers get bigger." Is the student right? Is this mathematically significant or interesting?
- Are there mathematically significant notions that underlie division of fractions?



Knowledge of content and curriculum (KCC)

- At what grade level are students typically taught to divide fractions?
- How is division of fractions related to division of whole numbers in the school curriculum?
- What are the models for fractions and for division with which students would be familiar?

Warranting the theory of MKT

The work of “proof” in research on the math that teachers need to know

What could warrant a claim about the mathematics that teachers should know?

- Opinion
- Professional judgment
- The content of the school curriculum
- Evidence about connection to—
  - mathematical quality of teaching
  - student learning

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- Opinion
- Professional judgment
- The content of the school curriculum
- Evidence about connection to—
  - mathematical quality of teaching
  - student learning

Empirical testing and improvement of the theory of MKT

1. Developing measures of MKT
2. Factor analyses and validation studies
3. Theory development

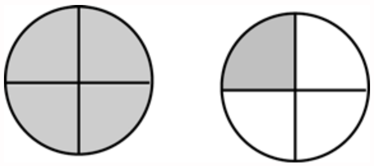
Analyzing non-standard (but correct) responses

Which student is using a method that could be used to multiply any two whole numbers?

Student A	Student B	Student C
35	35	35
x 25	x25	x 25
1 25	175	25
+ 7 5	+ 700	1 50
8 7 5	875	1 00
		+ 600
		8 7 5

Representing number concepts

Mrs. Johnson thinks it is important to vary the whole when she teaches fractions. For example, she might use five dollars to be the whole, or ten students, or a single rectangle. On one particular day, she uses as the whole a picture of two pizzas. What fraction of the two pizzas is she illustrating below? (Mark ONE answer.)



a) 5/4

b) 5/3

c) 5/8

d) 1/4

Providing mathematical explanations: Number concepts

Ms. Harris was working with her class on divisibility rules. She told her class that a number is divisible by 4 if and only if the last two digits of the number are divisible by 4. One of her students asked her why the rule for 4 worked. She asked the other students if they could come up with a reason, and several possible reasons were proposed. Which of the following statements comes closest to explaining the reason for the divisibility rule for 4? (Mark ONE answer.)

a) Four is an even number, and odd numbers are not divisible by even numbers.

b) The number 100 is divisible by 4 (and also 1000, 10,000, etc.).

c) Every other even number is divisible by 4, for example, 24 and 28 but not 26.

d) It only works when the sum of the last two digits is an even number.

Using data to test and improve theory

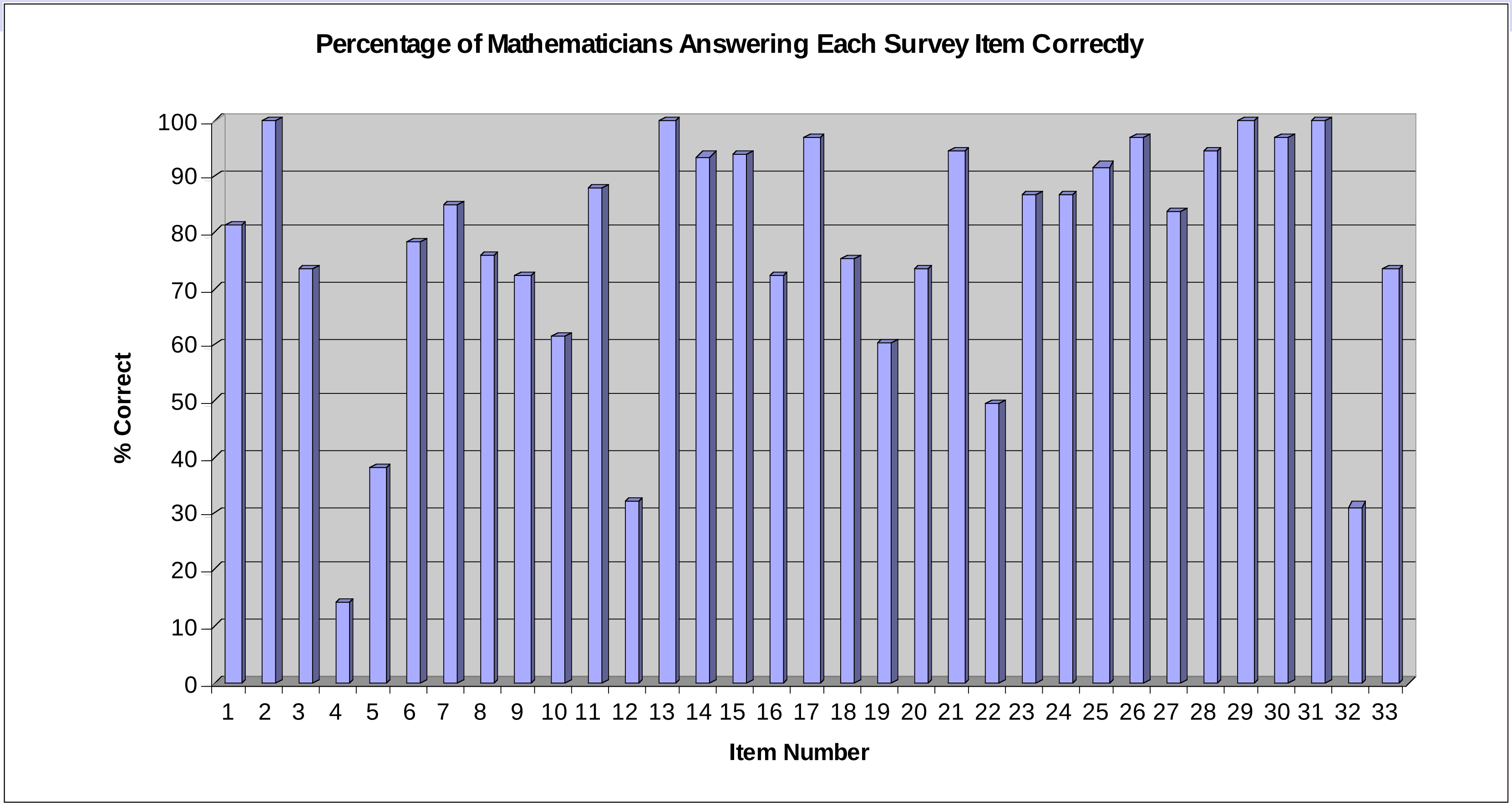
- Factor analyses
- Analyses of validity
- Uses of measures
  - To predict student achievement
  - To evaluate professional development

Validating MKT measures

How do we interpret teachers' performance on our questions?

1. Their score reflects their mathematical thinking
   – Cognitive interviews
2. Higher scores mean higher-quality mathematics instruction
   – Videotape validation study
3. Scores reflect common and specialized knowledge of content
   – Mathematician and non-teacher interviews
4. Higher scores related to improved student learning
   – Student gains analysis

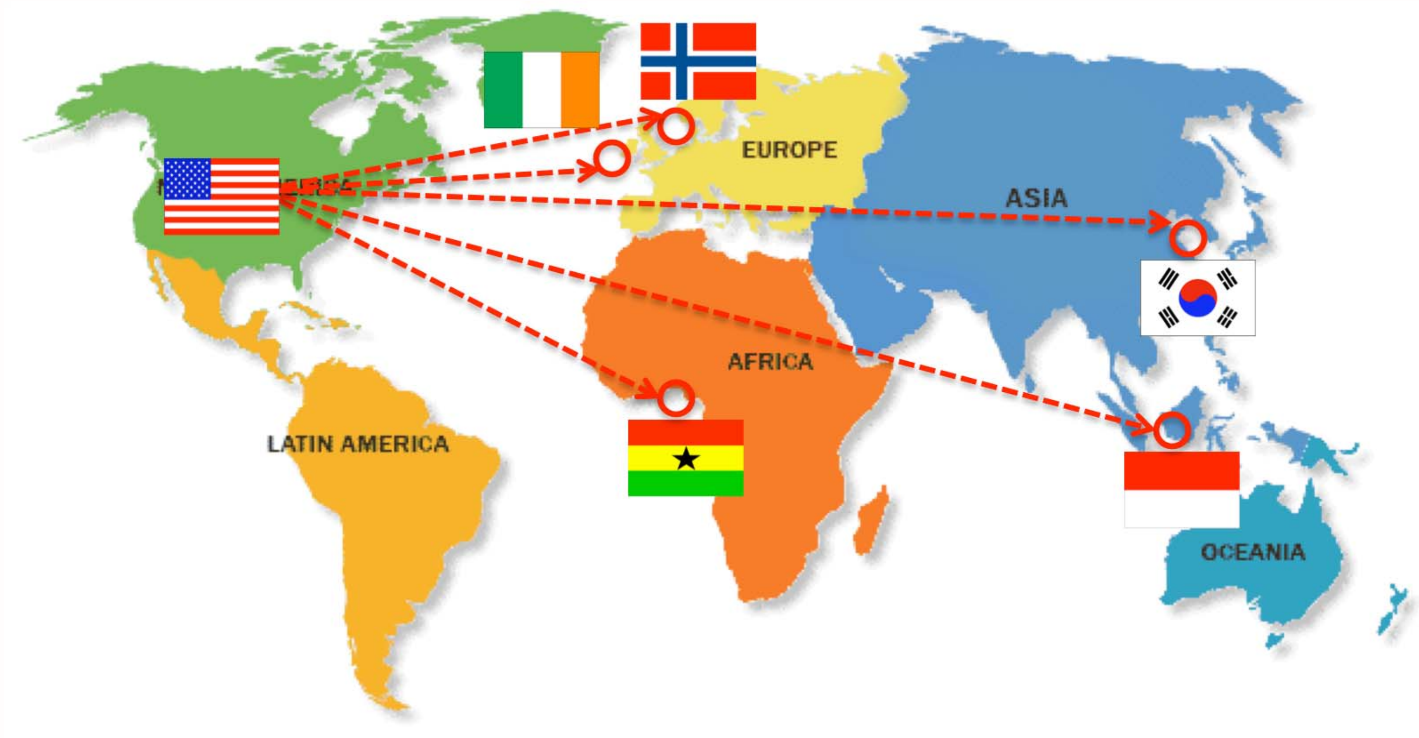
Percentage of mathematicians answering each survey item correctly



Why did mathematicians get items wrong?

- Items had mathematical flaws.
- Items required knowledge of learners.
- Items demanded mathematical knowledge unique to the work of teaching:
  - Making sense of non-standard solutions or ideas
  - Choosing numerical examples
  - Choosing representations

Are U.S.-based measures of MKT valid for research in other countries?



How can teachers best be taught the mathematics they need to know?

Approaches to developing teachers' MKT

1. Teach teachers what they need to teach (the school curriculum)
2. Teach teachers higher mathematical ideas connected to the school curriculum
3. Teach the topics of the school curriculum from an advanced perspective
4. Teach higher mathematics to provide broader perspective on the field

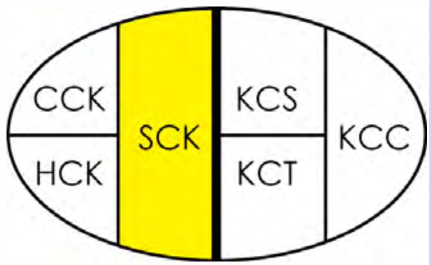
A challenge for teachers — and teacher educators

Teaching is a practice.

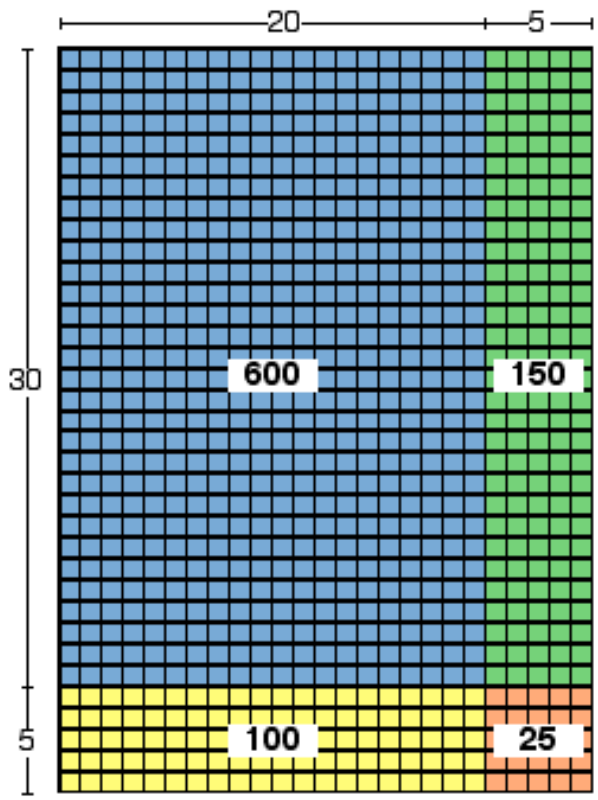
It is the *use of mathematics in practice* that matters.

Practicing the mathematical work of teaching

- Working on examples like we have done, including practicing providing reasons
- Developing fluency and speed
- Developing sensibilities about mathematical language
- Unpacking and situating mathematical ideas



Practice #1: Representing and mapping across representations



A

```
   35
 x 25
 ----
  125
 +75
 ----
  875
```

B

```
   35
 x 25
 ----
  175
+700
 ----
  875
```

C

```
   35
 x 25
 ----
   25
  100
  150
+ 600
 ----
  875
```

Practice #2: Becoming mathematically agile on your feet

$$\begin{array}{r} 27 \\ 38 \\ +\ 19 \\ \hline 74 \end{array}$$

What problem would you pose next to this pupil, and why?

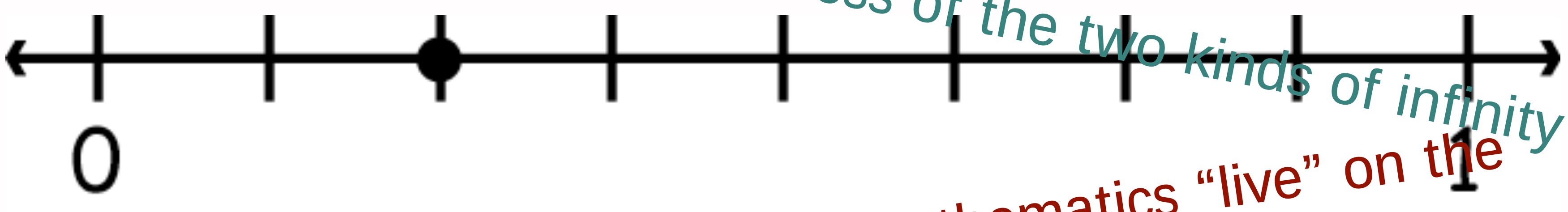
Practice #3: Developing sensibilities about mathematical language

- Use of quantifying terms: e.g., exactly, no more than, no less than, at least, at most
- Attention to the wording of mathematical tasks
- Care with definitions, their role, their requirements, and judgments about when, where, and why they are essential
- Awareness of the overlaps and conflicts between mathematical language and everyday language, other school language, and students' home languages

Unpacking and situating mathematical ideas

Exploring ideas of density of the rational numbers

Developing awareness of the two kinds of infinity



All the numbers of primary school mathematics “live” on the number line

Conclusion

1. Teaching involves doing mathematics in some special ways that are connected to the purposes of teaching practice — i.e., helping others learn mathematics.
2. This special kind of work involves special kinds of problems, engaging in special kinds of mathematical reasoning, and using mathematical language in specially careful ways.
3. Teachers need opportunities to develop skills with this special kind of mathematical work.

THANK YOU!

Slides will be available
at Deborah Ball's website

(Google "Deborah Ball")

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