Addressing the Mathematical Needs of Teacher Education: What can be offered by the ICMI-IMU Felix Klein Project ?

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Summary

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- The Klein project
  - Why the Klein project?
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- Focusing on a specific mathematics concept: the concept of function
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  - Some important changes since the time of Felix Klein
  - How can the Klein project contribute to make these changes accessible and inspiring for secondary teachers?
- Coming back to teacher education

The Felix Klein Project

Why the Felix Klein Project?

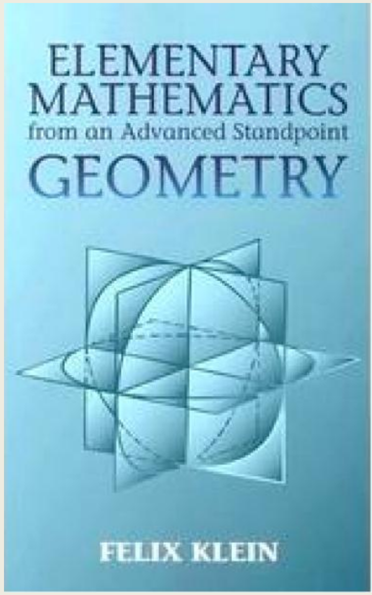
- The centennial of ICMI in 2008, the associated reflection on the activities of the Commission since its creation and the challenges it currently faces.

- Among these challenges, the challenge of teacher education.

- The relevance of the reflection developed by Felix Klein, one century ago in his lectures to high school teachers and the associated books "« Elementary mathematics from an advanced standpoint »

Felix Klein's motivation

"For a long time, university men were concerned exclusively with their sciences, without giving a thought to the needs of the schools, without even caring to establish a connection with school mathematics."



According to him, this attitude resulted in a double discontinuity.

Felix Klein’s ambition

Helping overcome these obstacles:

• paying specific attention to the connections between mathematical domains and to applications:

“My task will always be to show you the mutual connection between problems in the various fields. In this way, I hope to make it easier for you to acquire the ability to draw from the great body of knowledge there put before you a living stimulus for your teaching.”

• rejecting the formal approach dominating at that time in high school education and promoting what is known at that time as the “*genetic method*”.

How he sees his role:

“This book is designed as such a mental spur, not as a detailed handbook.”

A discourse meeting current concerns

- The disconnection between taught mathematics and the current state of this scientific field, the effect of such a disconnection on the social image of mathematics and on the attractiveness of mathematics careers.
- Teaching methods which very often do not allow the majority of students to understand the “raison d’être” of the mathematics they are taught and make sense of them, even when they are not so formal.
- But also:
  - an evolution of mathematical sciences both in terms of domains and practices, a development of internal and external connections which makes the task much more difficult and imposes necessary selection (Lovasz 2008).
  - an educational context which is radically different: students, teachers, and also the knowledge accumulated by educational research

The project organization and development

International Mathematical Union (IMU)

International Commission on Mathematical Instruction

- The project ambition: to make important developments of mathematics since the time of Felix Klein accessible to high school mathematics teachers in a way empowering them as professionals.
- A project requiring a tight connection between mathematics and mathematics education perspectives, thus a joint project of ICMI and IMU.
- A design group of 8 members chaired by Bill Barton, the current ICMI President, in charge of conceptualizing the project and organizing its realization.

Our initial vision of the Klein project

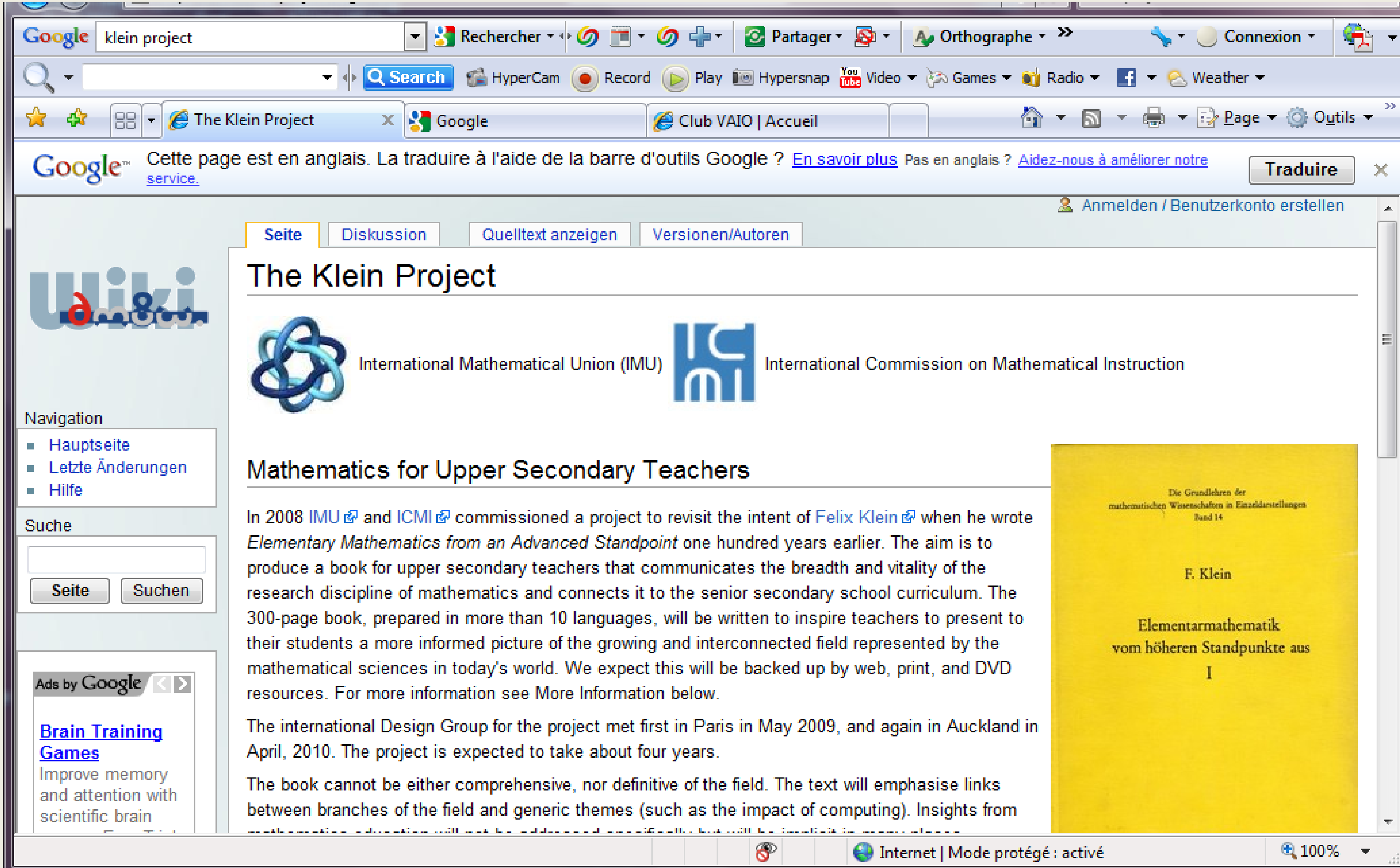
- A synthetic and stimulating book (about 300 pages), simultaneously published in different languages, addressed to senior high school mathematics teachers in priority (thus to mathematics teachers with some university mathematics background), and supposed interest in mathematics.
- A website complementing the book and including associated educational resources, open to contributions.
- A series of Klein conferences where mathematicians, mathematics educators, teacher educators and teachers would collaboratively work on the project, comment on drafts, offer their own contributions.

The book / The Wiki

The book:

- What is aimed at is not an advanced course but a simulating description, emphasizing the main ideas and ilustrating these with examples which can be inspiring for a secondary teacher.
- A structure including:
  - chapters focusing both on classical mathematical domains and on newer areas of increasing importance: logics, arithmetic, algebra and structures, geometry, functions and analysis, probabilities and statistics, discrete mathematics, computation
  - and also transversal chapters focusing on internal and external connections, and the ways mathematicians work.
- The impact of the first Klein conference in Madeira: the creation of the wiki “kleinproject.org” and the emergence of the idea of “Klein Vignette”.

www.kleinproject.org



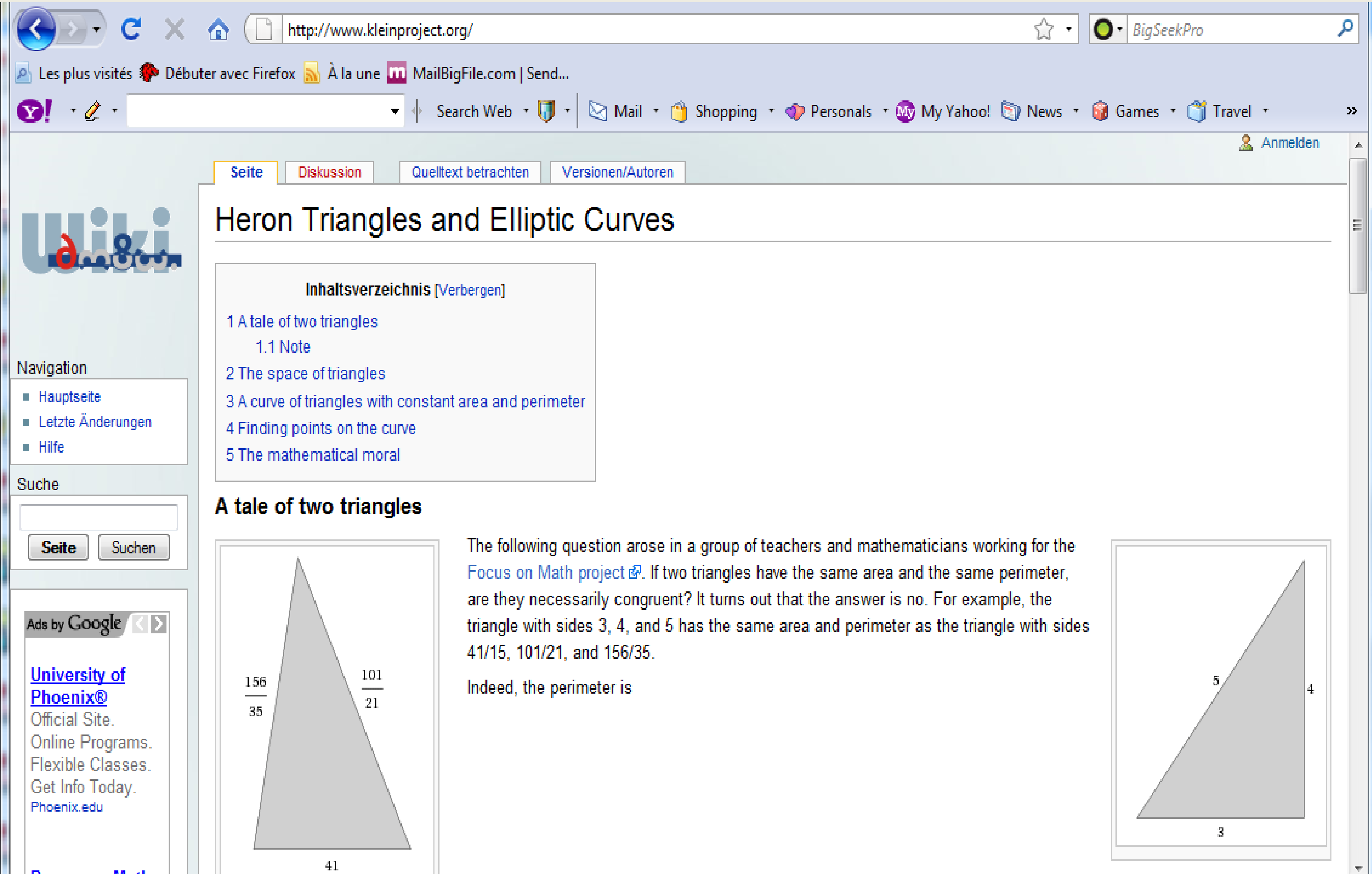
Klein Vignettes

- The idea of Klein Vignette: to illustrate an important mathematical idea or to present an important application of mathematics in a way potentially stimulating and inspiring for a secondary mathematics teacher.
- At the moment, two main types of vignettes:
  - vignettes proposing trajectories connecting school mathematics and more advanced and recent mathematics;
  - vignettes which make visible and understandable the mathematics implemented in artifacts or techniques.

Towards an identity for the Klein vignettes

- To be short (4 to 6 pages).
- To begin with an example or a problem stimulating for a secondary mathematics teacher.
- To involve recent mathematical developments (20th or 21st centuries).
- To explicitly show mathematics while limiting technical developments.
- To make explicit why the theme treated is important and to end by an explicit mathematical morale.
- To provide references easily accessible, for instance on line, allowing the interested teacher to deepen his/her mathematical reflection or envisage an educational exploitation of the vignette

The first vignette



Seite | Diskussion | Quelltext betrachten | Versionen/Autoren

Heron Triangles and Elliptic Curves

Inhaltsverzeichnis [Verbergen]

A tale of two triangles

The following question arose in a group of teachers and mathematicians working for the Focus on Math project. If two triangles have the same area and the same perimeter, are they necessarily congruent? It turns out that the answer is no. For example, the triangle with sides 3, 4, and 5 has the same area and perimeter as the triangle with sides 41/15, 101/21, and 156/35.

Indeed, the perimeter is

where $s = \frac{1}{2}(a+b+c)$ is the semiperimeter of the triangle. A quick calculation using this formula shows that the area of the triangle on the left is also 6.

Note

For a description of how the teachers and mathematicians in Focus on Math worked on the this question see Steven Rosenberg, Michael Spillane, and Daniel~B. Wulf, *Delving deeper: Heron triangles and moduli spaces*, Mathematics Teacher **101** (2008), no.~9, 656.

The space of triangles

How do we find examples like this? The secret is to find the right way of representing the space of all triangles. There are many possible ways to do this. For example, we could think of the space of triangles as the subspace of all triples $(a, b, c) \in \mathbb{R}^3$ corresponding to the three sides of the triangle. Not every point in $\mathbb{R}^3$ corresponds to a triangle; for example, all the coordinates must be positive. Can you think of other restrictions?

There's another way of putting coordinates on the space of triangles using angles instead of lengths. Every triangle has an inscribed circle, and the radius r of the circle has a simple relationship with the area A and semiperimeter s, namely

$$A = rs. \qquad (1)$$

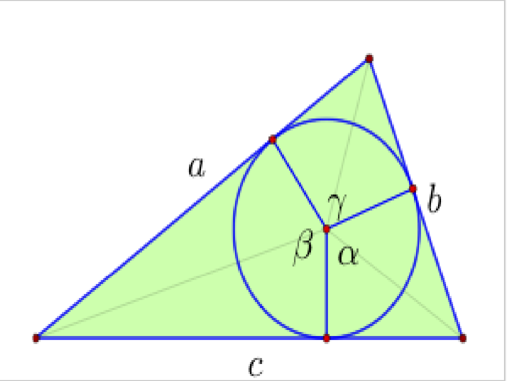


Figure 1:The three angles α, β, and γ determine a triangle.

To see why this is true, drop perpendiculars from the center of the circle to the sides of the triangle, as in Figure 1. These perpendiculars form the altitudes of 3 smaller triangles with bases on the sides of of the big triangle and and vertices at the center of the inscribed circle. Adding up the areas of these triangles we get equation (1).

Equation (1) tells us that if two triangles have the same area and same semiperimeter, then the radii of their inscribed circles are also the same. So if we are looking for two such triangles we will find them in the space of all triangles inscribed around a fixed circle. Instead of using lengths to parameterize this space, we will use the angles formed by the three radii at the center of incircle, as in in Figure 1.

A curve of triangles with constant area and perimeter

of the incircle, as follows. The radii and the lines from the vertices to the incenter break the triangle into six right triangles. Because the lines from the vertices to the center bisect the angles of the big triangle, these right triangles occur in congruent pairs. Taking one base length from each pair and adding , we get

$$s = r(\tan\frac{\alpha}{2} + \tan\frac{\beta}{2} + \tan\frac{\gamma}{2}). \qquad (2)$$

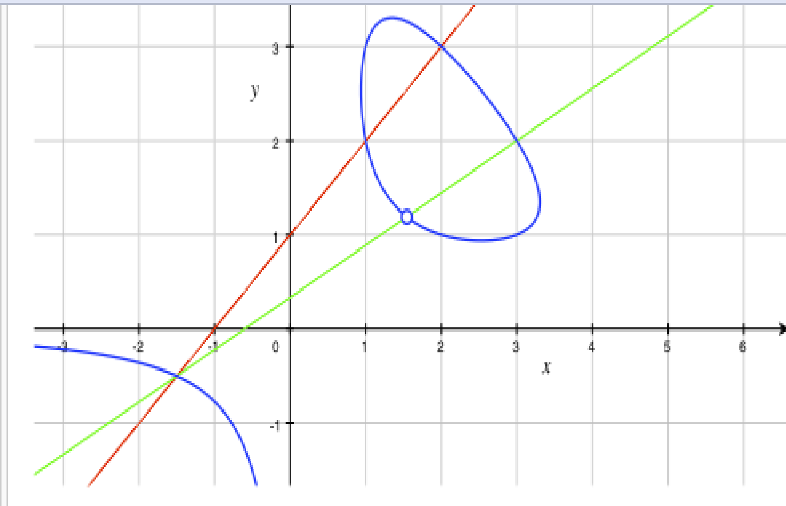


Figure 2: the curve of triangles

Equations (1) and (2) together tell us that if the area A and semiperimeter s are constant, then so is the sum of the tangents:

$$\tan\frac{\alpha}{2} + \tan\frac{\beta}{2} + \tan\frac{\gamma}{2} = \frac{s^2}{A}. \qquad (3)$$

Second, we translate this condition into an equation defining a curve in the plane. Let $x = \tan(\alpha/2)$, $y = \tan(\beta/2)$, and $z = \tan(\gamma/2)$. Since $\alpha + \beta + \gamma = 2\pi$, we have

$$\frac{\gamma}{2} = \pi - \frac{\alpha}{2} - \frac{\beta}{2},$$

so

$$z = \tan\left(\frac{\gamma}{2}\right) = \tan\left(\pi - \frac{\alpha}{2} - \frac{\beta}{2}\right) = -\tan\left(\frac{\alpha}{2} + \frac{\beta}{2}\right) = -\frac{x+y}{1-xy}.$$

Then, if k is the constant s^2/A, equation (3) becomes for fixed k, the equation

$$x + y - \frac{x+y}{1-xy} = k,$$

The mathematical moral

“The study of elliptic curves is a central area of research in number theory, with applications to the cryptographic schemes behind secure financial transactions on the web. Elliptic curves played a central role in the proof of Fermat's Last Theorem.

The story described in this article shows the remarkable unity of mathematics, starting as it does in high school and ending in research. Along the way we encountered a fundamental idea in modern mathematics: the idea of solving a problem about a particular type of object (triangles with area 6 and perimeter 12, for example) by situating the object in a more general space (the space of all triangles) and finding the right way of parametrizing that space. “

The Felix Klein Conferences

- Madeira, Portugal (October 2009), the emergence of the Klein project in Portuguese language
- Castro Urdale, Spain (June 2010)
- Oxford, England (June 2010)
- Belo-Horizonte, Brasil (July 2010)
- Pittsburg Mathfest, USA (August 2010)
- Stockholm, Sweden (June 2011)
- Palo Alto, USA (November 2011)

Focusing on a specific concept:

The concept of function

The concept of function: a key concept for Felix Klein

“We, who are called the reformers, would put the function concept at the very center of instruction, because, of all the concepts of the mathematics of the past two centuries, this one plays the leading role wherever mathematical thought is used. We would introduce it into instruction as early as possible with constant use of graphical method, the representation of functional relations in the x y system, which is used today as a matter of course in every practical application of mathematics.” (p.4)

The Felix Klein’s vision

- An historical reference to the dual vision of the idea of function present in Euler’s work, and the subsequent evolution leading:
  - on the one hand to the theory of analytical functions by Lagrange,
  - on the other hand, to the general definition of functions in terms of arbitrary correspondence expressed by Dirichlet in relation to the study of Fourier’s series.
- The attention paid to the definition of transcendental functions *(“the most important thing for us to discuss”)* and to the representation of functions by trigonometric series.
- A genetic vision of teaching, following the historical development with some hysteresis, using inductive and heuristic approaches based on perception.

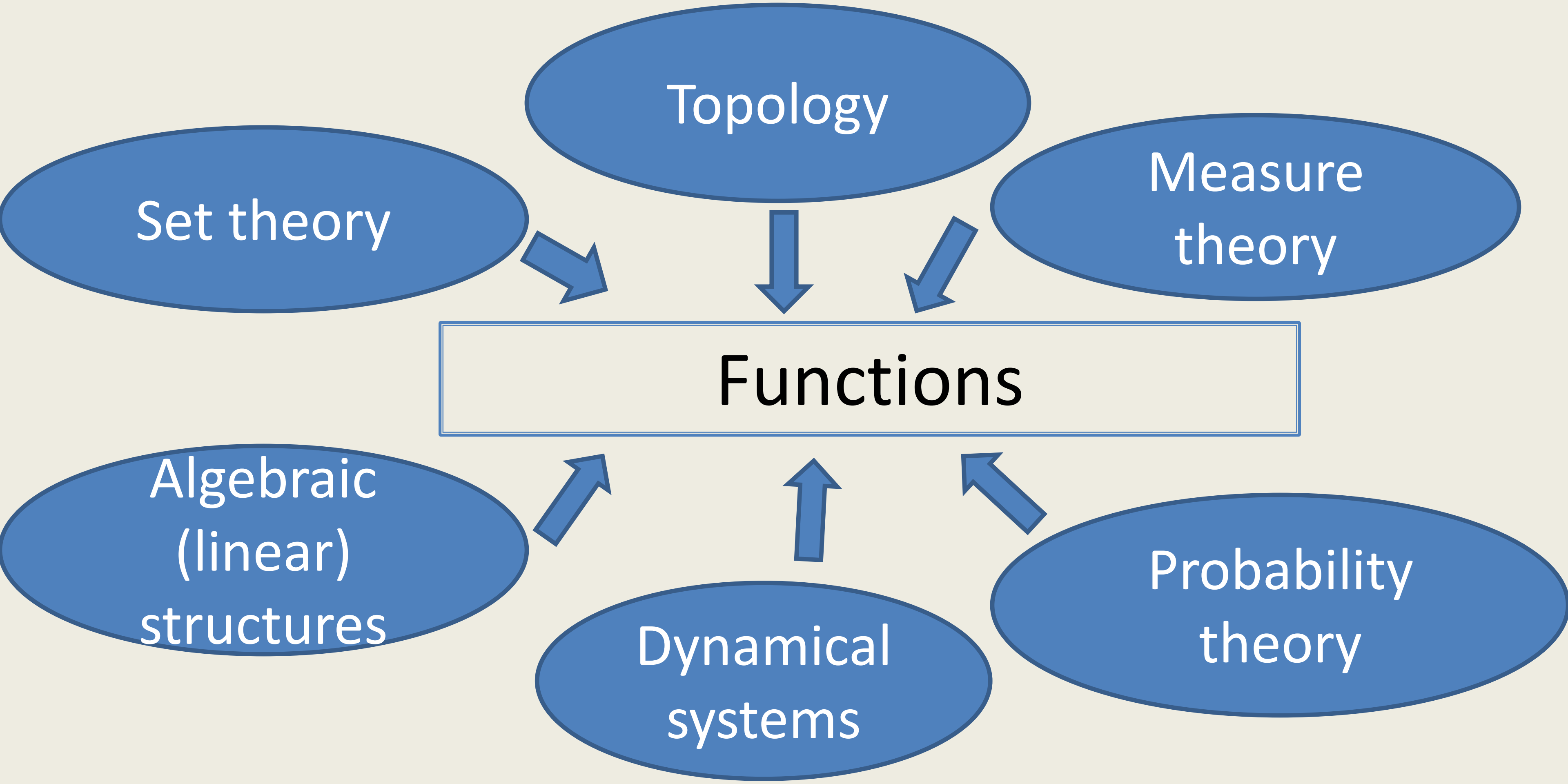
The Felix Klein's vision

"We desire merely that the general notion of function, according to the one or the other of Euler interpretations, should permeate as a ferment the entire mathematical instruction in the higher schools. It should not, of course, be introduced by means of abstract definitions, but should be transmitted to the students as a living possession, by means of elementary examples, such as one finds in large number in Euler." (p.205)

The Felix Klein’s vision

- But also, a rather critical vision of recent developments and interest for functions “which contain the most disagreeable singularities balled into horrid lumps”.
- And associated, a distinction made between two types of generalizations:
  - those being developed in reference to applications,
  - those which “are the result purely of the love of mathematical research, which has taken no account whatever of the needs of natural phenomena, and the results have indeed found as yet no direct application.”

Since Klein's time: a huge and multidimensional evolution



With important consequences

- On the vision of this domain:
  - new perspectives on the notion of function,
  - new perspectives on the relationships between normality and pathology,
  - a renewed vision of the “linear” (local/global)
  - new connections
- On mathematical practices, essentially due to the technological evolution.

How to convey this evolution in the Klein project?

The necessity to take into account:

- the evolution of teachers' preparation and culture,
- the advances of didactical knowledge regarding this particular theme, but also more generally teachers'knowledge, representations and practices,
- the evolution of educational contexts, of modes of access to information, of social practices, the diversity of resources currently available.

How to select inspiring themes?

- **The epistemological potential**: considering the mathematical ideas and meta-ideas that the theme makes it possible to illustrate and work on.
- **The didactical potential:** considering the potential for addressing important curricular issues, bridge with secondary mathematics.
- **The practice potential**: considering the mathematical practices that the theme potentially engages, makes it possible to illustrate.
- **The connection potential**: considering the connections that the theme can support between representations, areas, fields.
- **The challenging potential**: considering the potential offered for questioning common views, stimulating interest, linking mathematics with the outside world
- **The cultural potential**: considering the potential offered to reflect mathematics as a cultural and historical enterprise, as well as a living science.

One example: The fixed point theorem of Banach

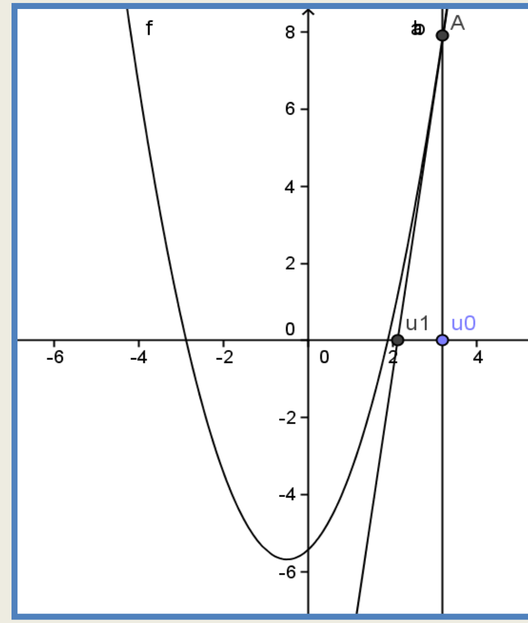
- A theorem central in Analysis in order to ensure the existence of objects and find ways for approximating them.
- A theorem bridging qualitative and quantitative work in Analysis in the most modern ways.
- A theorem already present, at least implicitly, in secondary education, in the particular context of functions of one variable, but whose extension to functional spaces shows the power of Analysis.
- A theorem having applications en various areas, both internal and external to mathematics.
- A theorem which makes it possible to revisit and understand differently mathematical situations already familiar, and also to establish connections between very ancient mathematics and current developments of the discipline.

A formulation of the theorem

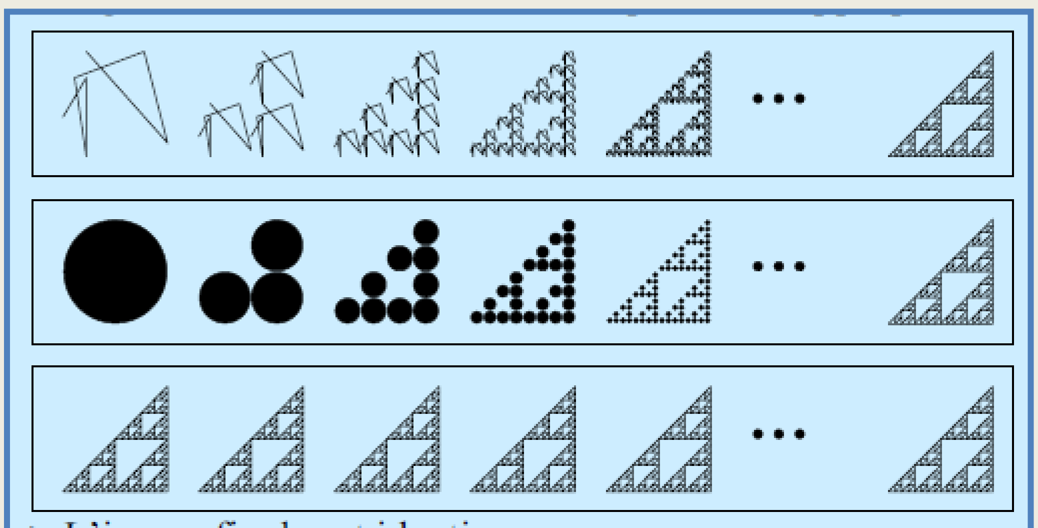
If f is a contracting mapping in a complete metric space, then f has a unique fixed point, and any sequence defined by the recurrence relation $u_{n+1}=f(u_n)$ converges towards this fixed point whatever is its initial term.

Bridging history and current mathematics

The so-called
Heron’s sequence
$X_{n+1}=0.5(x_n+a/x_n)$



The iterative generation of fractals



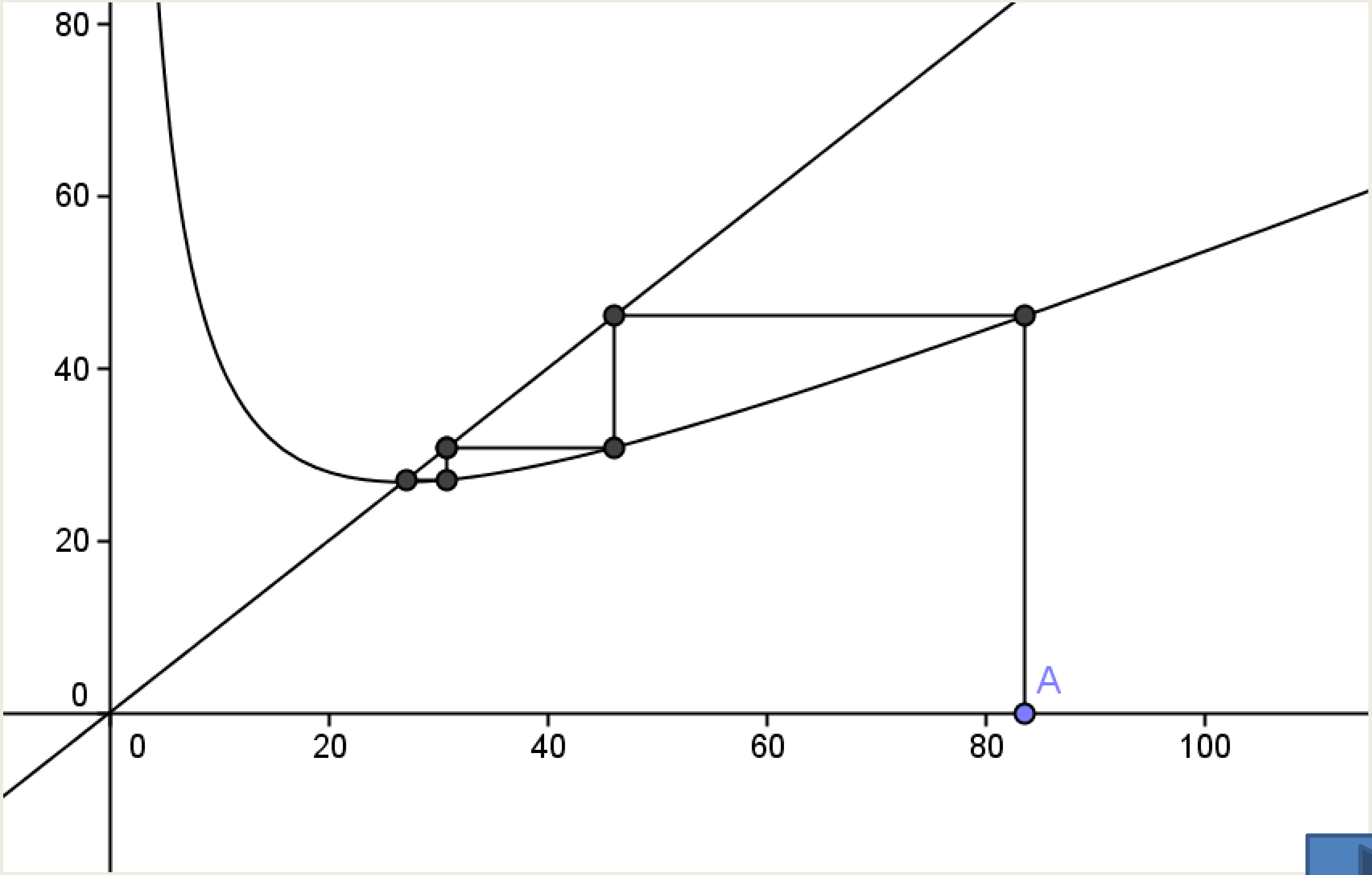
The Newton’s method implemented in hand calculators

The Heron's method

'Since', says Heron,[1] '720 has not its side rational, we can obtain its side within a very small difference as follows. Since the next succeeding square number is 729, which has 27 for its side, divide 720 by 27. This gives $26\frac{2}{3}$. Add 27 to this, making $53\frac{2}{3}$, and take half of this or $26\frac{1}{2}\frac{1}{3}$. The side of 720 will therefore be very nearly $26\frac{1}{2}\frac{1}{3}$. In fact, if we multiply $26\frac{1}{2}\frac{1}{3}$ by itself, the product is $720\frac{1}{36}$, so that the difference (in the square) is $\frac{1}{36}$.

'If we desire to make the difference still smaller than $\frac{1}{36}$, we shall take $720\frac{1}{36}$ instead of 729 [or rather we should take $26\frac{1}{2}\frac{1}{3}$ instead of 27], and by proceeding in the same way we shall find that the resulting difference is much less than $\frac{1}{36}$.'

Vizualizing the Heron's sequence



The connection with Newton's method

$f(x)=x^2 - 720$

Equation of the tangent at the point $(x_n, f(x_n))$:

$y-f(x_n) = 2x_n (x-x_n)$

Intersection with (Ox):

$x_{n+1} = x_n - f(x_n)/2x_n$

$x_{n+1} = (x_n + 720/x_n)/2$

A fixed point super-attractive

More generally: $x_{n+1} = x_n - f(x_n)/f'(x_n)$

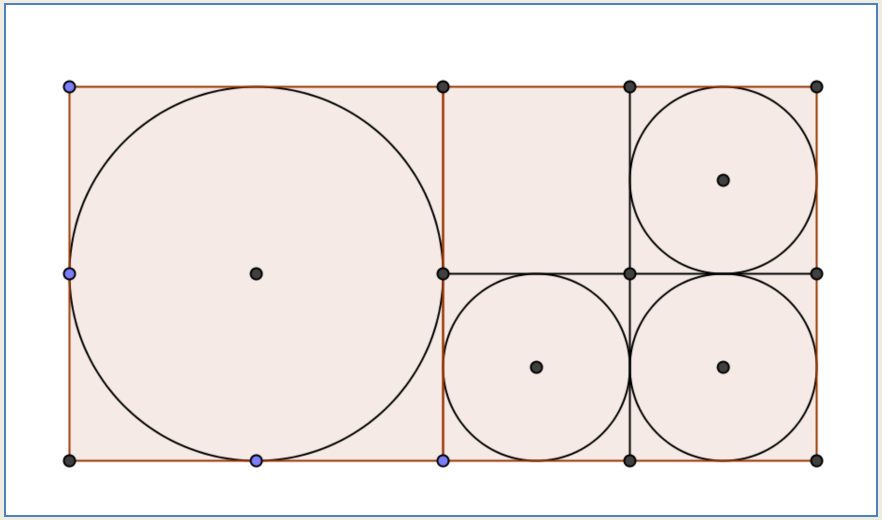
$x_{n+1} = g(x_n)$ with $g(x) = x - f(x)/f'(x)$

$$g'(x) = 1 - 1 + f(x).f''(x)/[f'(x)]^2$$

If $f(a)=0$ then $g'(a)=0$

Generating fractal sets

Three afine transformations

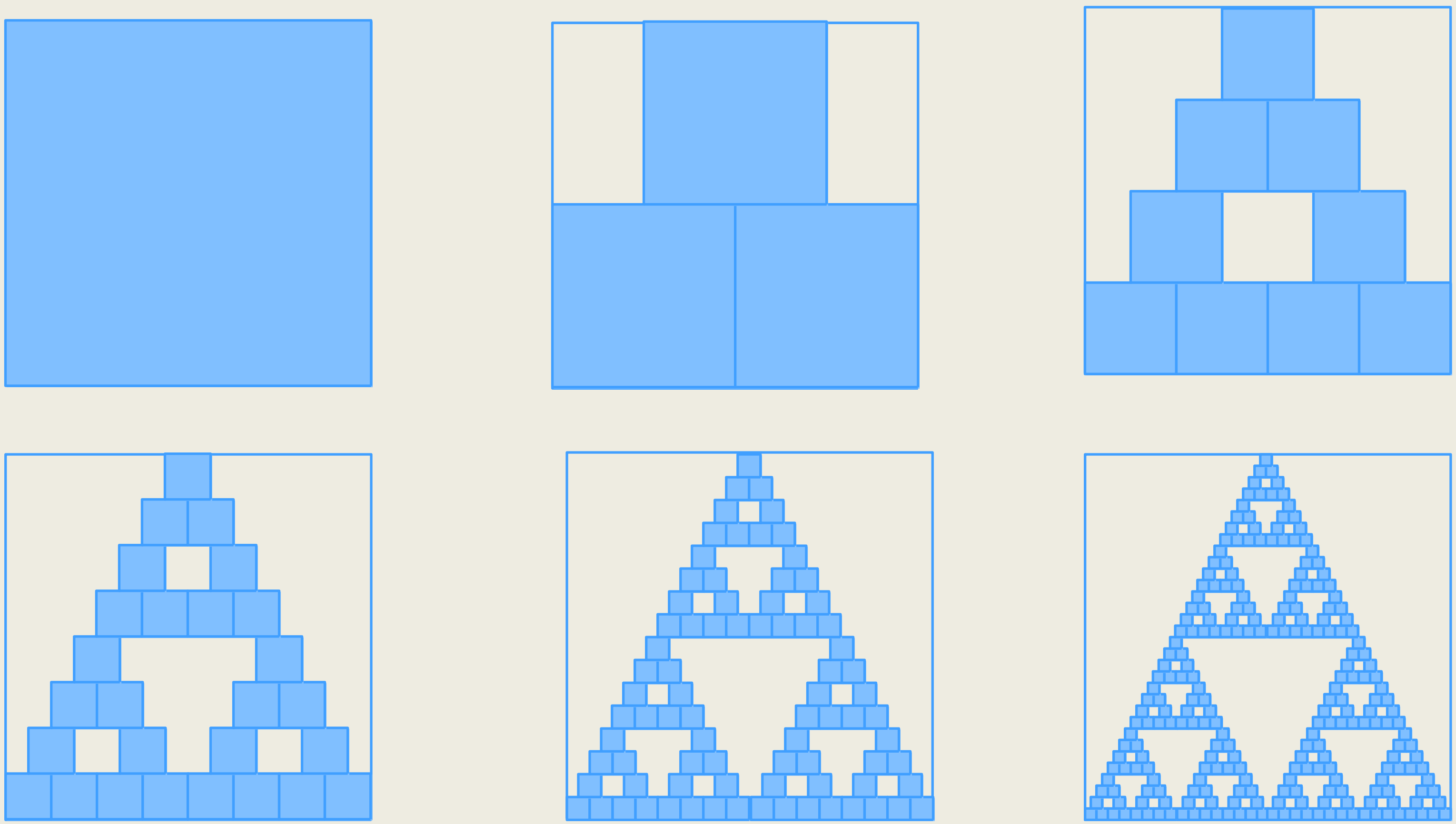


$(x,y) \Rightarrow (x/2 + 1/2\ ,\ y/2 + 1/2)$

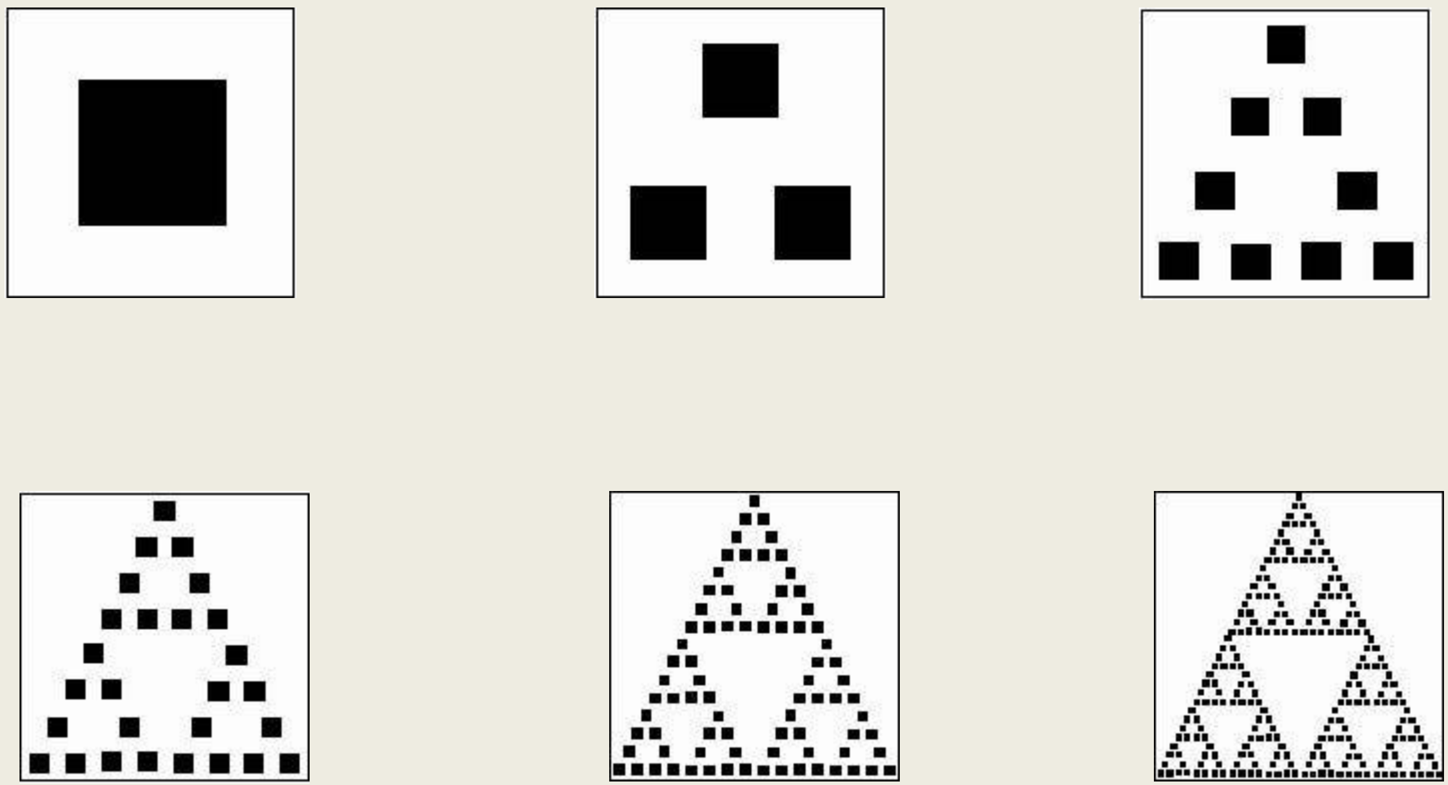
$(x,y) \Rightarrow (x/2 + 1/2\ ,\ y/2 - 1/2)$

$(x,y) \Rightarrow (x/2 - 1/2,\ y/2 - 1/2)$

Another close transformation



With another initial figure: apparently the same limit



Application to the production and compression of images

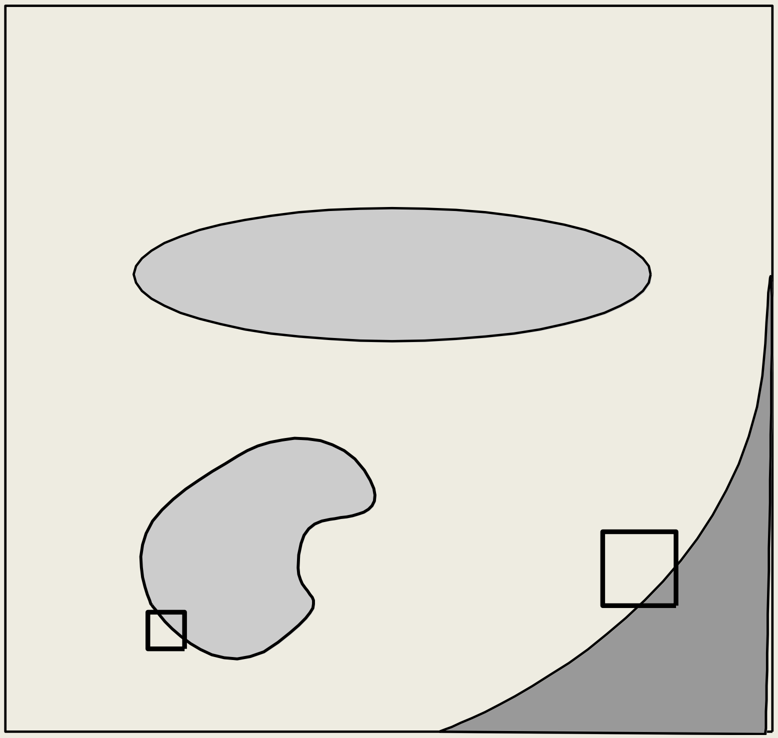
The production of a fern

(Rousseau & Saint Aubin, 2008)

The corresponding program in Mathematica

```
m=15000
L[n_]:=If[1<n<87,2,n]
H[n_]:=If[86<n<94,3,L[n]]
K[n_]:=If[n>93,4,H[n]]
R=Table[K[Random[Integer,{1,100}]],{m}];
F[1,x_,y_]:=0
G[1,x_,y_]:=0.16*y
F[2,x_,y_]:=x*0.85+y*0.04
G[2,x_,y_]:=-x*0.04+y*0.85+1.6
F[3,x_,y_]:=x*0.2-y*0.26
G[3,x_,y_]:=0.23*x+0.22*y+1.6
F[4,x_,y_]:=-x*0.15+y*0.28
G[4,x_,y_]:=x*0.26+y*0.24+0.44
x[1]:=0
y[1]:=0
Do[{x[n+1],y[n+1]}={F[R[[n]],x[n],y[n]],G[R[[n]],x[n],y[n]]},{n,1,m}]
T=Table[{x[n],y[n]},{n,m}];
ListPlot[T, AspectRatio->1, Axes-> False]
```

Compression of images



Coding:

We replace any small square by the image of a similar larger square under a homothety of ratio ½ composed with one of 8 transformations:

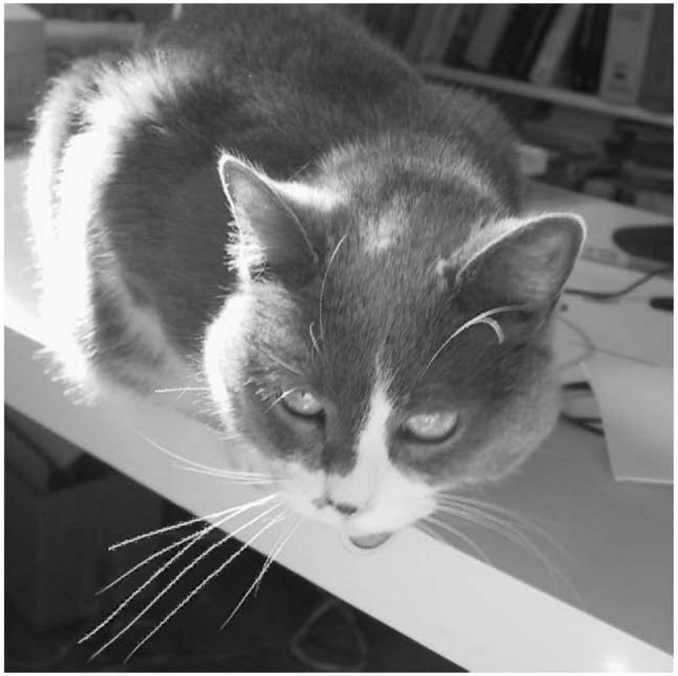
- Identity plus 3 rotations
- 4 symetries

We adjust contrast.

We make a translation of the level of grey.

An example

First iteration

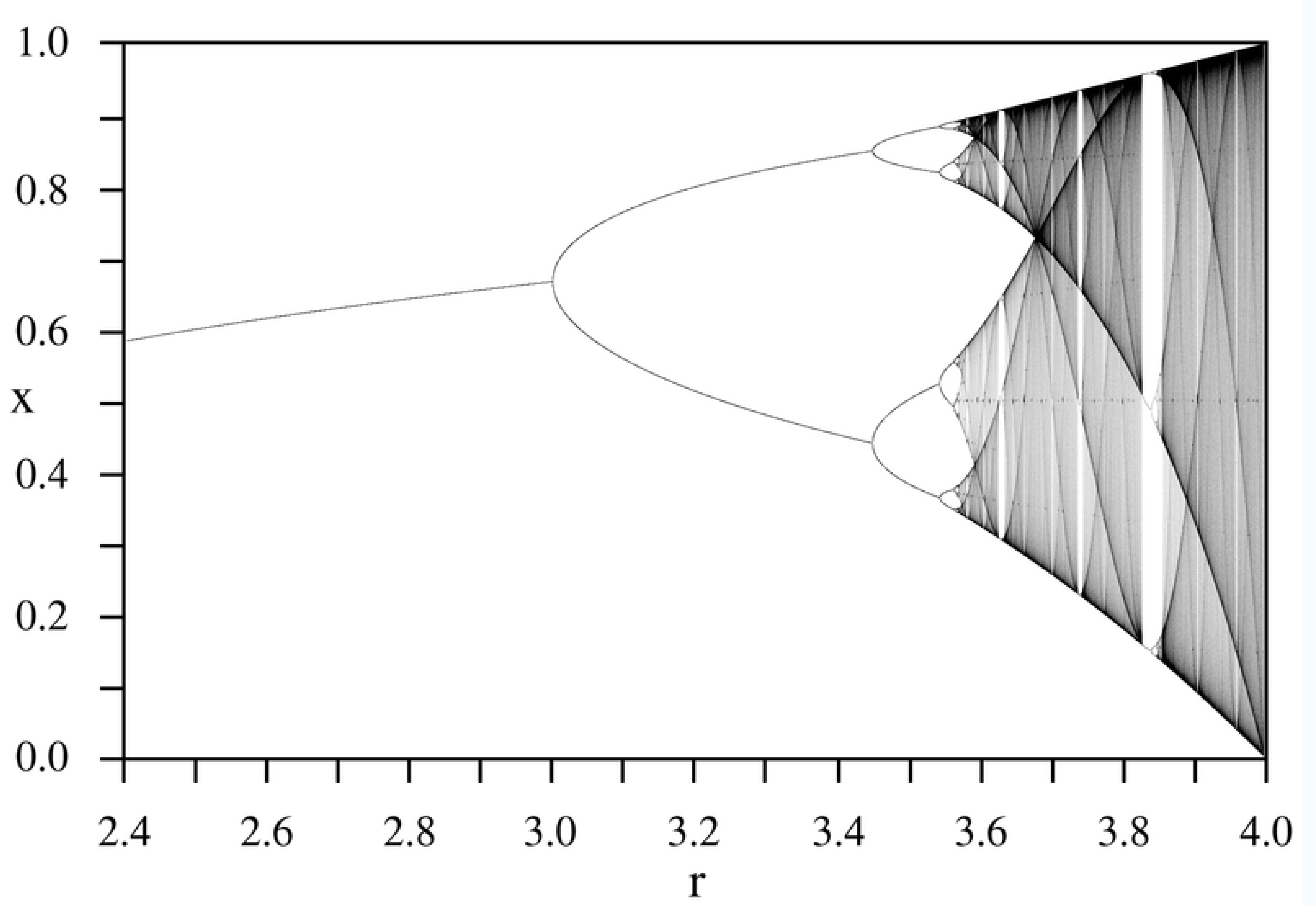


Sixth iteration

Coming back to teacher education

An example of use in teacher education

1. Starting with the Heron's method for approaching square roots.
2. Studying it with high school mahematical tools both qualitatively and quantitatively, using CAS.
3. Revisiting this method at the light of fixed point theory, including the distinction between attractive and super-attractive fixed points.
4. Connecting with Newton's method.
5. Reinvesting in the study of logistic sequences, meeting bifurcation phenomenas and transition to chaos (video + detailed paper).
6. Generalizing the view on the fixed point theorem, considering more general spaces, connecting the arguments used. Adding an algorithmic perspective.


1.0
0.8
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x
0.4
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2.8
3.0
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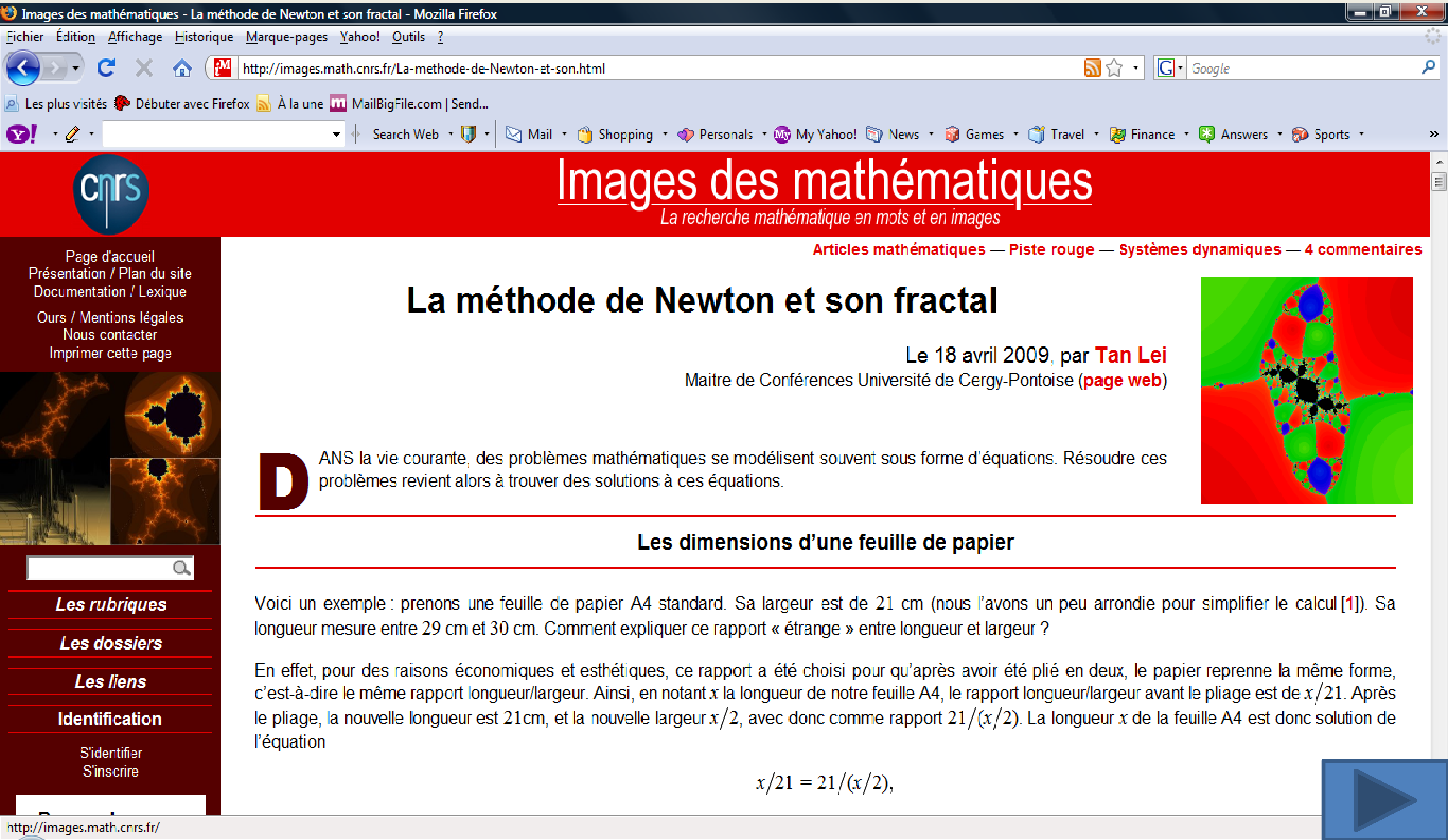
4.0
r

http://images.math.cnrs.fr/La-methode-de-Newton-et-son.html



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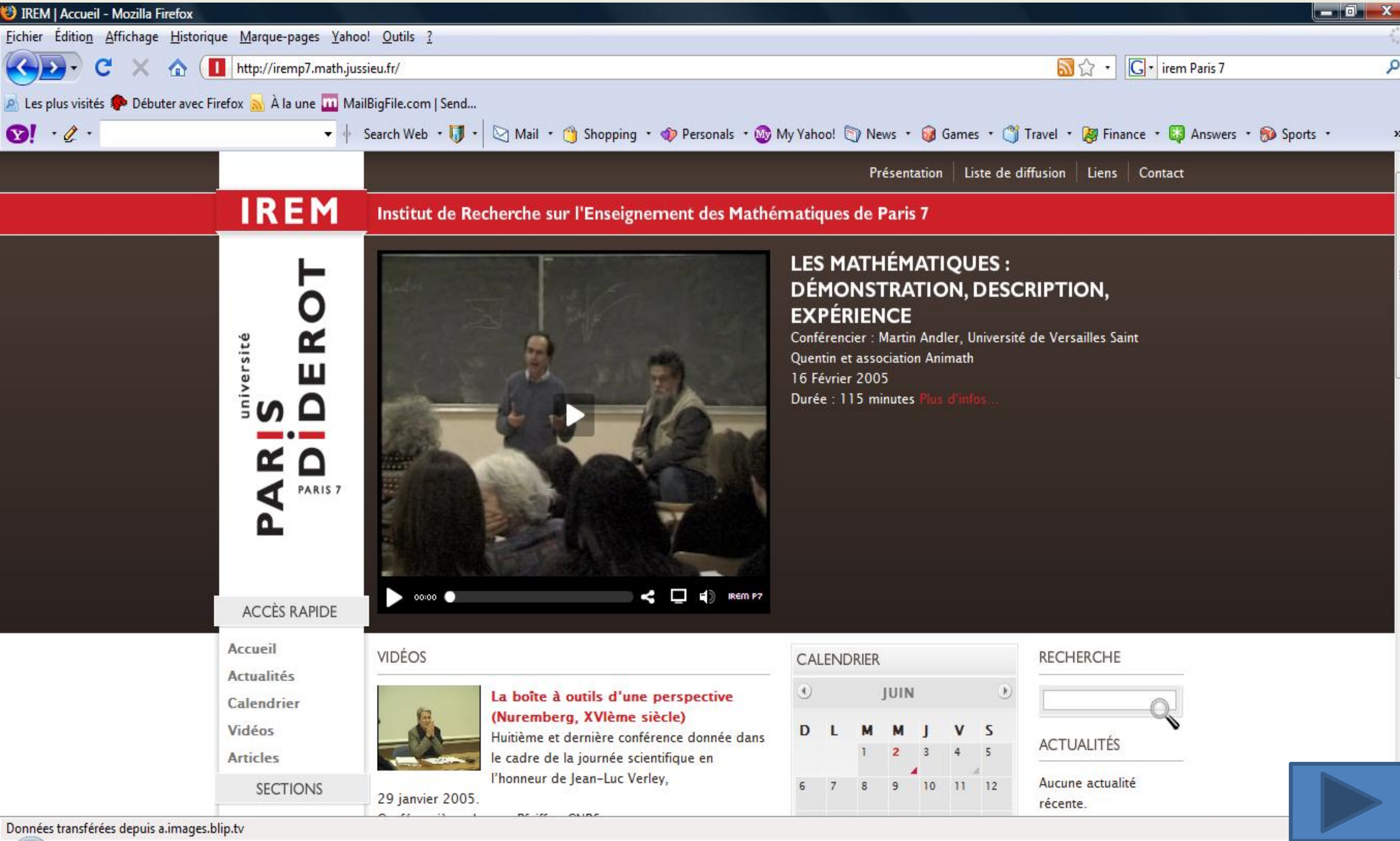

Teacher education: some legitimate ambitions

- **Consolidation and connections**: ensuring a deep understanding of elementary analysis, including the awareness of the specific modes of reasoning in that domain and reasonable technical operationality, taking into account the support provided by technology, emphasizing connections with other domains and with secondary maths,
- **Development**: introducing to more advanced perspectives on rather familiar objects, taking into account the evolution of the field.
- **Openess:** meeting new mathematical objects and perspectives, phenomena today highly popularized…

And, as a member of the Klein project

My wish and that of the design team that the reflection you are developing regarding teacher knowledge will generate contributions to the Felix Klein Project